SPIN

AND

ANGULAR MOMENTUM

2ND LECTURE
FROM THE COURSE
QUANTUM PHYSICS OF LOW DIMENSIONAL STRUCTURES

QPLDS

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GLIWICE

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A SPIN AND AN ANGULAR MOMENTUM

Angular momentum is a quantity which satisfy following equation

$$[J_i, J_k] = i\hbar \varepsilon_{ikl} J_l, \quad i,k,l = 1,2,3$$

where

$$\varepsilon_{ikl} = \begin{cases} 1, i \neq k \neq l \text{ and the ikl can be set in growing order after even permutations of the every two subscripts} \\ -1, \quad i \neq k \neq l \text{ and the ikl can be set in growing order after odd permutations of the every two subscripts} \\ 0, \text{if at least two subscripts are equal} \end{cases}$$

and the angular momentum for the spin has two eigen-values: $\pm \frac{\hbar}{2}$

Another type of the electron-spin operator

$$J^{(s)} = \frac{1}{2}\hbar\sigma$$

This has also two eigen-values: ± 1, and

$$[\sigma_i, \sigma_k] = i\hbar \varepsilon_{ikl} \sigma_l, \quad i,k,l = 1,2,3$$

Properties of the σ-Pauli matrixes:

a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

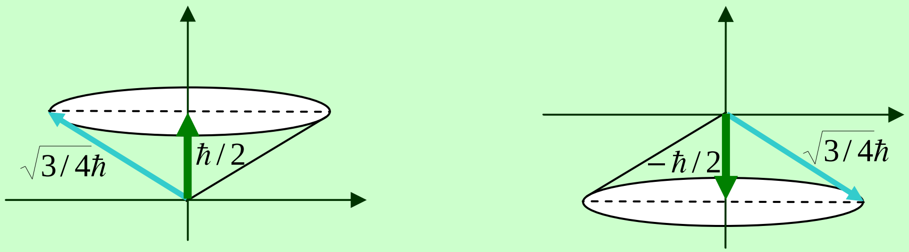
b) $Tr[\sigma]=0$

c) $\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \sigma_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \quad \sigma_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

d) $\left[\sigma^2,\sigma_i\right]=0$

This is why

$$(J^{(s)})^2=\frac{\hbar^2}{4}(\sigma_x^2+\sigma_y^2+\sigma_z^2)=\frac{3}{4}\hbar^2 I \Rightarrow \left|J^{(s)}\right|=\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\hbar\ ,\ m_s=\pm\frac{1}{2}$$



Looking for eigen-vectors of the σ_z

$$\sigma_z\kappa=\lambda\kappa$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = +1, \lambda_2 = -1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = +\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{cases} a = a \\ -b = b \end{cases} \Rightarrow b = 0, \quad \kappa_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{cases} a = -a \\ b = b \end{cases} \Rightarrow a = 0, \quad \kappa_2 = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

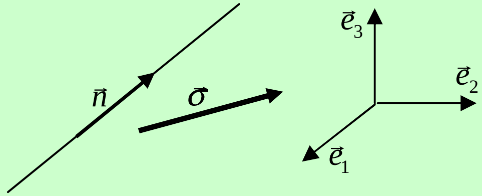
Normalized vectors

$$\kappa_1 = \begin{bmatrix} a=1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_1 = 1 \qquad \text{and} \qquad \kappa_2 = \begin{bmatrix} 0 \\ b=1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1$$

vectors κ_1 and κ_2 are orthogonal $\kappa_i \kappa_k = \delta_{ik}$ as

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

PROJECTION OF A SPIN ONTO AN ARBITRARY DIRECTION



$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_2 \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3$$

$$\boxed{\vec{\sigma} = \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3}$$

$$\vec{\sigma} = \begin{bmatrix} \vec{e}_3 & \vec{e}_1 - i\vec{e}_2 \\ \vec{e}_1 + i\vec{e}_2 & -\vec{e}_3 \end{bmatrix}$$

There is no eigen-solution for the spin oriented arbitrarily

$$\vec{\sigma}\chi = \vec{n}\chi$$

but there is for the spin-projection eigen-problem

$$\boxed{(\vec{\sigma}\,\vec{n})\chi = \lambda\chi}$$

The proof

$$\vec{\sigma}\,\vec{n} = \begin{bmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{bmatrix}, \quad n_+ = n_1 + in_2, \; n_- = n_1 - in_2$$

$$(\vec{\sigma}\,\vec{n})\chi = \lambda\chi$$

$$\begin{bmatrix} n_3 & n_- \\ n_+ & -n_3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{cases} n_3 a + n_- b = \lambda a \\ n_+ a - n_3 b = \lambda b \end{cases} \Rightarrow \begin{vmatrix} n_3 - \lambda & n_- \\ n_+ & -n_3 - \lambda \end{vmatrix} = 0$$

$$(n_3 - \lambda)(-n_3 - \lambda) - n_+ n_- = 0 \;\Rightarrow\; \lambda^2 = n_3^2 + n_+ n_- = n_1^2 + n_2^2 + n_3^2 = 1$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 \;\Rightarrow\; b_1 = -\frac{n_3 - 1}{n_-} a_1 \quad b_2 = -\frac{n_3 + 1}{n_-} a_2$$

now, a little trick, as the arbitrary direction $\vec{n} = [n_1, n_2, n_3]$ can be easily described in the spherical coordinates

$$n_1 = \sin\theta\cos\varphi, \quad n_2 = \sin\theta\sin\varphi, \quad n_3 = \cos\theta$$

then,

$$n_{\pm} = n_1 \pm i n_2 = \sin\theta\cos\varphi \pm i\sin\theta\sin\varphi = \sin\theta(\cos\varphi \pm i\sin\varphi) = \sin\theta e^{\pm i\varphi}$$

$$b_1 = -\frac{\cos\theta - 1}{\sin\theta e^{-i\varphi}} a_1 = -\frac{\cos\theta - 1}{\sin\theta e^{-i\varphi}} a_1 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)e^{-i\varphi}} a_1 = e^{i\varphi} tg\left(\frac{\theta}{2}\right) a_1$$

$$b_2 = -\frac{\cos\theta + 1}{\sin\theta e^{-i\varphi}} a_2 = -e^{i\varphi} ctg\left(\frac{\theta}{2}\right) a_2$$

and the eigen-functions expressed in the $\begin{bmatrix} a \\ b \end{bmatrix}$ form are equal to

$$\chi_1 = \begin{bmatrix} 1 \\ e^{i\varphi} tg\left(\frac{\theta}{2}\right) \end{bmatrix} a_1 \quad a_1 = \cos(\theta/2) e^{-i\varphi/2} \quad \Rightarrow \quad \chi_1 = \begin{bmatrix} e^{-i\varphi/2} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi/2} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\chi_2 = \begin{bmatrix} 1 \\ -e^{i\varphi} ctg\left(\frac{\theta}{2}\right) \end{bmatrix} a_2 \quad a_2 = -\sin(\theta/2) e^{-i\varphi/2} \quad \Rightarrow \quad \chi_2 = \begin{bmatrix} -e^{-i\varphi/2} \sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi/2} \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\boxed{\chi_i^+ \chi_k = \delta_{ik}}$$

Superposition of the two spins

The target: to solve eigen problem for the resultant spin composed from the two single-particle spins

$\chi_1(1) = \chi_{1/2}(1)$ - the eigen-function of the first particle with the spin-projection equals $+\hbar/2$

$\chi_2(1) = \chi_{-1/2}(1)$ - the eigen-function of the first particle with the spin-projection equals $-\hbar/2$

$\chi_1(2) = \chi_{1/2}(2)$ - the eigen-function of the second particle with the spin-projection equals $+\hbar/2$

$\chi_2(2) = \chi_{-1/2}(2)$ - the eigen-function of the second particle with the spin-projection equals $-\hbar/2$

Next, the problem can be solved using 2-component products as this is two-body problem now

The starting (input) physical-state vectors are

$\chi_{1/2}(1)\chi_{1/2}(2) \quad \chi_{1/2}(1)\chi_{-1/2}(2) \quad \chi_{-1/2}(1)\chi_{1/2}(2) \quad \chi_{-1/2}(1)\chi_{-1/2}(2)$

The operator of the resultant spin is a simple sum of the single-particle operators

$$\boxed{\hat{J}^{(s)} = \frac{\hbar}{2}[\hat{\sigma}(1) + \hat{\sigma}(2)]}$$

$$(\hat{J}^{(s)})^2 = \frac{\hbar^2}{4}[3 + \hat{\sigma}(1)\hat{\sigma}(2)]$$

$$\hat{\sigma}(1)\hat{\sigma}(2) \quad !$$

Let’s roll…

From the four input function we can compose 3 symmetrical and one anti-symmetrical functions:

symmetrical

$$\chi_{1/2}(1)\chi_{1/2}(2),\ \chi_{-1/2}(1)\chi_{-1/2}(2),$$

$$\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)+\chi_{-1/2}(1)\chi_{1/2}(2)]$$

anti-symmetrical

$$\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)-\chi_{-1/2}(1)\chi_{1/2}(2)]$$

Let's check how these functions are influenced by the $\hat{\sigma}(1)\hat{\sigma}(2)$ operator

$$[\hat{\sigma}(1)\hat{\sigma}(2)]\chi_{1/2}(1)\chi_{1/2}(2)=1\cdot\chi_{1/2}(1)\chi_{1/2}(2)$$

$$[\hat{\sigma}(1)\hat{\sigma}(2)]\chi_{-1/2}(1)\chi_{-1/2}(2)=1\cdot\chi_{-1/2}(1)\chi_{-1/2}(2)$$

$$[\hat{\sigma}(1)\hat{\sigma}(2)][\chi_{1/2}(1)\chi_{-1/2}(2)+\chi_{-1/2}(1)\chi_{1/2}(2)]=$$

$$=1\cdot[\chi_{1/2}(1)\chi_{-1/2}(2)+\chi_{-1/2}(1)\chi_{1/2}(2)]$$

$$[\hat{\sigma}(1)\hat{\sigma}(2)][\chi_{1/2}(1)\chi_{-1/2}(2)-\chi_{-1/2}(1)\chi_{1/2}(2)]=$$

$$=-3\cdot[\chi_{1/2}(1)\chi_{-1/2}(2)-\chi_{-1/2}(1)\chi_{1/2}(2)]$$

These results should be putted into followings

$$(\hat{J}^{(s)})^2 = \frac{\hbar^2}{4}[3+\hat{\sigma}(1)\hat{\sigma}(2)],\ (\hat{J}^{(s)})^2\chi(1)\chi(2) = \lambda\cdot\chi(1)\chi(2),\quad \lambda = \hbar^2 S(S+1)$$

in order to obtain

$$(\hat{J}^{(s)})^2 = \frac{\hbar^2}{4}[3+\hat{\sigma}(1)\hat{\sigma}(2)] = \begin{cases}\hbar^2\cdot S(S+1),\ \textit{for the symmetrical case}\\ \hbar^2\cdot 0,\ \textit{for the antisymmetrical case}\end{cases}$$

Thus, we conclude:

$$\boxed{(\hat{J}^{(s)})^2 = S(S+1)\hbar^2 \quad \begin{cases}S=1,\ \textit{a triplet state}\\ S=0,\ \textit{a singlet state}\end{cases}}$$

However, for the z-component only we have

$$\frac{\hbar}{2}[\sigma_3(1)+\sigma_3(2)]\chi_{1/2}(1)\chi_{1/2}(2) = +1\cdot\hbar\chi_{1/2}(1)\chi_{1/2}(2)$$

$$\frac{\hbar}{2}[\sigma_3(1)+\sigma_3(2)]\chi_{-1/2}(1)\chi_{-1/2}(2) = -1\cdot\hbar\chi_{1/2}(1)\chi_{1/2}(2)$$

$$\frac{\hbar}{2}[\sigma_3(1)+\sigma_3(2)]\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)+\chi_{-1/2}(1)\chi_{1/2}(2)] = 0$$

$$\frac{\hbar}{2}[\sigma_3(1)+\sigma_3(2)]\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)-\chi_{-1/2}(1)\chi_{1/2}(2)] = 0$$

THE WAVE-EQUATION FOR A PARTICLE POSSESING A SPIN IN THE PRESENCE OF EXTERNALLY APPLIED MAGNETIC FIELD

Simple approach

$$H^{(s)} = -\mu_0 \vec{\sigma} \cdot \vec{H}$$

The spin-orbit coupling is excluded

Let $H^{(0)}$, $\psi^{(0)}$, E_0 (Hamiltonian, wave-function, energy) describe the electron move with spin excluded

Let $H^{(s)}$, χ, E_s describe only interaction of the spin with the external field

$$H^{(0)}\psi^{(0)} = E_0 \psi_0$$

$$H^{(s)}\chi = E_s \chi$$

$$(H^{(0)} + H^{(s)})\psi^{(0)}\chi = (E_0 + E_s)\psi_0 \chi$$

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix} = a\chi_{1/2} + b\chi_{-1/2}, \quad |a|^2 + |b|^2 = 1$$

$$\psi^{(0)}\chi = a\psi^{(0)}\chi_{1/2} + b\psi^{(0)}\chi_{-1/2}$$

$$\int \psi_\mu^* \chi_k^+ \psi_\nu \chi_m dV = \delta_{\mu\nu}\delta_{km}$$

$$\int \psi_\mu^* \chi_k^+ A \psi_\nu \chi_m dV = \delta_{\mu\nu}\delta_{km} < A >$$

BUT as $|a|^2 + |b|^2 = 1$

THEN $< A >_s \delta_{km} = \chi_k^+ A \chi_m$ (no integral ! pure space)

More realistic approach

a) $H = H(\vec{r}, \vec{p}, \vec{\sigma}, t)$

b) $\psi = \psi(\vec{r}, s, t)$

c) $\psi = \psi(\vec{r}, s, t) = \begin{bmatrix} \psi_{1/2}(\vec{r}, t) \\ \psi_{-1/2}(\vec{r}, t) \end{bmatrix}, \quad \psi^+(\vec{r}, s, t) = \begin{bmatrix} \psi^*_{1/2}(\vec{r}, t) & \psi^*_{-1/2}(\vec{r}, t) \end{bmatrix}$

the eigen-equation

$$-\frac{\hbar}{i}\frac{\partial \psi(\vec{r}, s, t)}{\partial t} = H(\vec{r}, \vec{p}, \vec{\sigma}, t)\psi(\vec{r}, s, t)$$

Let the electron be in a rest state (stationary case)

$$-\frac{\hbar}{i}\frac{\partial \psi(\vec{r}, s, t)}{\partial t} = -\mu_0 \vec{\sigma} \cdot \vec{H}\, \psi(\vec{r}, s, t)$$

It's time now to make a trick with time

$$\psi = \psi(\vec{r}, s, t) = \psi(\vec{r}, s) e^{-iEt/\hbar}$$

to separate time timely from the problem for a moment J

$$-\mu_0\vec{\sigma}\cdot\vec{H}\,\psi(\vec{r},s)=E\,\psi(\vec{r},s)$$

and for the externally applied magnetic field oriented along the z-direction we have

$$-\mu_0\sigma_z H\,\psi(\vec{r},s)=E\,\psi(\vec{r},s)$$

$$-\mu_0 H\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\psi_{1/2}\\ \psi_{-1/2}\end{bmatrix}=E\begin{bmatrix}\psi_{1/2}\\ \psi_{-1/2}\end{bmatrix}$$

Thus the eigen-values are equal to

$$E_{1/2}=-\mu_0 H \qquad E_{-1/2}=\mu_0 H$$

with the eigen-states expressed as

$$\psi_{1/2}=\begin{bmatrix}1\\ 0\end{bmatrix} \qquad \psi_{-1/2}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

but for the “time back-included” we have

$$\psi_{1/2}=\begin{bmatrix}1\\ 0\end{bmatrix}e^{-\mu_0 Ht/\hbar} \qquad \psi_{-1/2}=\begin{bmatrix}0\\ 1\end{bmatrix}e^{-\mu_0 Ht/\hbar}$$

For the arbitrary orientation of the magnetic field $\vec{H}=[H_1,H_2,H_3]$ we have

$$\begin{bmatrix} H_3 - E & H_- \\ H_+ & -H_3 - E \end{bmatrix} \begin{bmatrix} \psi_{1/2} \\ \psi_{-1/2} \end{bmatrix} = 0, \quad H_\pm = H_1 \pm iH_2$$

which is similar to the previously solved problem:

$$\begin{bmatrix} n_3 & n_- \\ n_+ & -n_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

Is that something to be memorized finally?

…we can compose 3 symmetrical and one anti-symmetrical functions:

symmetrical

$$\chi_{1/2}(1)\chi_{1/2}(2)$$

$$\chi_{-1/2}(1)\chi_{-1/2}(2)$$

$$\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)+\chi_{-1/2}(1)\chi_{1/2}(2)]$$

anti-symmetrical

$$\frac{\sqrt{2}}{2}[\chi_{1/2}(1)\chi_{-1/2}(2)-\chi_{-1/2}(1)\chi_{1/2}(2)]$$

It will be used later…