2^{ND} QUANTIZATION FORMALISM

QUANTUM OPTICS

7^{TH} LECTURE
FROM THE COURSE
QUANTUM PHYSICS OF LOW DIMENSIONAL STRUCTURES

QPLDS

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CONTENTS

1. Quantization of the EM field in the one-dimensional cavity
2. The vacuum state
3. Quantum fluctuations of the single-mode field
4. A hybrid laser – the example

QUANTUZATION OF RADIATION FIELD IN A ONE-DIMENSIONAL CAVITY

Coordinate system:

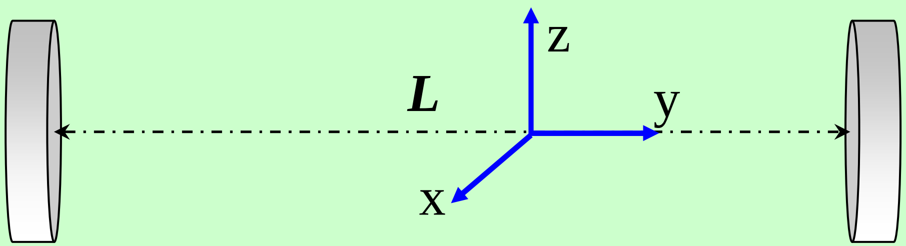
- cavity axis – "*y*"
- perpendicular (vertical) directions to the *y* axis – "*x* and *z*"

from Maxwell equations in a vacuum, with vanishing electric fields at the cavity boundaries, results the solution for the linearly polarized electric field vector (the standing wave)

$$E_z(y,t) = \sqrt{\frac{2m\omega^2}{V\varepsilon_0}} q(t) \sin(k \cdot y) \qquad E_z(y,t) = \sqrt{\frac{L}{V}} q(t) \sin(k \cdot y)$$

and for the magnetic field vector

$$H_x(y,t) = \frac{\varepsilon_0}{k} \sqrt{\frac{2m\omega^2}{V\varepsilon_0}} \frac{dq}{dt} \cos(k \cdot y) \qquad H_x(y,t) = \frac{\varepsilon_0}{k} \sqrt{\frac{L}{V}} \frac{dq}{dt} \cos(k \cdot y)$$



A DENSITY ENERGY OF A SINGLE-MODE ELECTROMAGNETIC FIELD equals

$$E_V = \frac{1}{2}(\varepsilon_0 E_z^2 + \mu_0 H_x^2) = \frac{1}{2}\left(\varepsilon_0 \frac{2m\omega^2}{V\varepsilon_0} q^2(t)\sin^2(ky) + \mu_0 \left(\frac{\varepsilon_0}{k}\right)^2 \frac{2m\omega^2}{V\varepsilon_0}\left(\frac{dq}{dt}\right)^2 \cos^2(ky) \right)$$

What is the mass *m* in the above?!

,…,

The cavity, shown in the Fig., is a type of oscillator device – by comparison to the harmonic potential energy the elastic constant of a spring ($\omega = \sqrt{k/m} \;\; \Rightarrow \;\; E_p = (1/2)k \cdot q^2(t) = (1/2)m\omega^2 \cdot q^2(t)$) $m\omega^2$ equals to

$$\boxed{m\omega^2 = \frac{1}{2}\varepsilon_0 L}$$

The $\frac{1}{2}\varepsilon_0 L$ measures the single-mode electromagnetic field “elasticity” of the cavity…

Do we lost main subject of this lecture for a moment…?

$$E_V = \frac{1}{2}(\varepsilon_0 E_z^2 + \mu_0 H_x^2) = \frac{1}{2}\left(\varepsilon_0 \frac{2m\omega^2}{V\varepsilon_0} q^2(t)\sin^2(ky) + \mu_0 \left(\frac{\varepsilon_0}{k}\right)^2 \frac{2m\omega^2}{V\varepsilon_0}\left(\frac{dq}{dt}\right)^2 \cos^2(ky) \right)$$

$$E_V = \frac{1}{V}\left(\varepsilon_0 \frac{m\omega^2}{\varepsilon_0} q^2(t)\overline{\sin^2(ky)} + \mu_0 \left(\frac{\varepsilon_0}{k}\right)^2 \frac{m\omega^2}{\varepsilon_0}\left(\frac{dq}{dt}\right)^2 \overline{\cos^2(ky)} \right)$$

$$E_V = \frac{1}{V}\left(\varepsilon_0 \frac{m\omega^2}{\varepsilon_0} q^2(t)\frac{1}{2} + \mu_0 \frac{\varepsilon_0^2}{\omega^2\varepsilon_0\mu_0} \frac{m\omega^2}{\varepsilon_0}\left(\frac{dq}{dt}\right)^2 \frac{1}{2} \right)$$

and finally

$$\boxed{E_V = \frac{1}{V}\left(\frac{1}{2} m\omega^2 q^2(t) + \frac{m^2}{m}\left(\frac{dq}{dt}\right)^2 \frac{1}{2} \right) = \frac{1}{V}\left(\frac{1}{2} m\omega^2 q^2(t) + \frac{p^2}{2m} \right)}$$

QUANTIZATION at least

$[\hat{q},\hat{p}]=i\hbar$ ($\hat{q},\hat{p}$ are hermitian operators)

$$E_V = \frac{1}{V}\left(\frac{1}{2}m\omega^2\hat{q}^2(t)+\frac{\hat{p}^2}{2m}\right)$$

and the Hamiltonian

$$H = \frac{1}{2}m\omega^2\hat{q}^2(t)+\frac{\hat{p}^2}{2m}$$

Introducing $[a,a^+]=1$ (a,a^+ are non-hermitian operators)

$$\hat{a} = \frac{1}{\sqrt{\frac{\varepsilon_0 L}{\omega}\hbar}}\left(\frac{\varepsilon_0 L}{\omega}\hat{q}+i\hat{p}\right) \qquad \hat{a}^+ = \frac{1}{\sqrt{\frac{\varepsilon_0 L}{\omega}\hbar}}\left(\frac{\varepsilon_0 L}{\omega}\hat{q}-i\hat{p}\right)$$

We get a Hamiltonian in the 2nd quantization formalism

$$\boxed{H = \hbar\omega\left(\hat{a}\hat{a}^+ + \frac{1}{2}\right)}$$

$$\boxed{a_k^+ a_k \,| n_k >= n_k \,| n_k >}$$

$$\hat{E}_z(y,t) = \sqrt{\frac{\hbar\omega}{V\varepsilon_0}}(\hat{a} + \hat{a}^+)\sin(k \cdot y)$$

$$H_x(y,t) = \frac{\varepsilon_0}{k}\sqrt{\frac{\hbar\omega}{V\varepsilon_0}}\left(\frac{d\hat{a}}{dt} + \frac{d\hat{a}^+}{dt}\right)\cos(k \cdot y)$$

So, we arrived at the problem: $d\hat{a}/dt$?!

HEISENBERG SPEAKS….

$$-\frac{\hbar}{i}\frac{d\hat{a}}{dt} = [H, \hat{a}]$$

$$[H, \hat{a}] = [\hbar\omega(\hat{a}^+\hat{a} + 1/2), \hat{a}] = -\hbar\omega\hat{a}$$

$$\frac{d\hat{a}}{dt} = -i\omega\hat{a}$$

similarly

$$\frac{d\hat{a}^+}{dt} = i\omega\hat{a}^+$$

and the solutions are

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \qquad \hat{a}^+(t) = \hat{a}(0)e^{i\omega t}$$

thus, the magnetic field vector equals

$$H_x(y,t) = i\omega \frac{\varepsilon_0}{k}\sqrt{\frac{\hbar\omega}{V\varepsilon_0}}\left(\hat{a}^+ - \hat{a}\right)\cos(k \cdot y)$$

DESCRIPTION OF A SINGLE-MODE FIELD ENERGY

$$H\,|\,n>=[\hbar\omega(\hat{a}^+\hat{a}+1/2)]\,|\,n>=E_n\,|\,n>$$

$$H\,|\,n>=[\hbar\omega(\hat{a}^+\hat{a}+1/2)]\,|\,n>=E_n\,|\,n> \quad /\,\hat{a}^+ \mapsto$$

$$[\hbar\omega(\hat{a}^+\hat{a}^+\hat{a}+1/2\hat{a}^+)]\,|\,n>=E_n\hat{a}^+\,|\,n>$$

$$\hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - 1$$

$$\hbar\omega[\hat{a}^+(\hat{a}\hat{a}^+ - 1) + 1/2\hat{a}^+]\,|\,n>=E_n\hat{a}^+\,|\,n>$$

$$\hbar\omega[\hat{a}^+\hat{a} + 1/2]\hat{a}^+\,|\,n>=(E_n + \hbar\omega)\hat{a}^+\,|\,n>$$

summarizing $H(\hat{a}^+\,|\,n>) = (E_n + \hbar\omega)(\hat{a}^+\,|\,n>)$

and similarly $H(\hat{a}\,|\,n>) = (E_n - \hbar\omega)(\hat{a}\,|\,n>)$

technical remark

as we can say that $\hat{a}^{+}|n>$ creates the $|n+1>$ state, however $\hat{a}^{+}|n>\neq|n+1>$, but $\hat{a}^{+}|n>=c_{n}^{*}|n+1>$ and $\hat{a}|n>=c_{n}|n-1>$

THE REMINDER ABOUT A VACUUM STATE FROM THE ***L3***

Let's imagine that we repeat following operation acting on different states

$$a_k|...k...>=(-1)^m|...\not{k}...>$$

and, after that, finally we get the vacuum state $|0>$

$$a_k^{+}|0>=|k>$$

for an arbitrary k

ALSO (for all k)

$$a_k|k>=|0>,\ a_k|0>=0,\ \text{and}\ <0|0>=1$$

The vacuum state must possess any, but positive energy

$$H(\hat{a}\,|\,0>) \neq (E_0 - \hbar\omega)(\hat{a}\,|\,0>)$$

thus

$$\hat{a}\,|\,0> \equiv 0$$

and

$$H\,|\,0> = \left[\hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)\right]|\,0> = \frac{1}{2}\hbar\omega\,|\,0>$$

coming back to these c_n and c_n^* in

$\hat{a}^+ | n >= c_n^* | n+1 >$ and $\hat{a} | n >= c_n | n-1 >$ ...

$\hat{a}^+ \| n >= c_n^* \| n+1 >$	$\hat{a} \| n >= c_n \| n-1 >$
$< n \| \hat{a}\hat{a}^+ \| n >=< n+1 \| c_n c_n^* \| n+1 >$ $< n \| \hat{a}\hat{a}^+ \| n >= \| c_n \|^2$ $< n \| \hat{a}^+\hat{a} + 1 \| n >= \| c_n \|^2$ $< n \| \hat{n} + 1 \| n >= \| c_n \|^2$ $< n \| \hat{n} \| n > + < n \| n >= \| c_n \|^2$ $n + 1 = \| c_n \|^2$ $\sqrt{n+1} = c_n^*$	$< n \| \hat{a}^+\hat{a} \| n >=< n-1 \| c_n^* c_n \| n-1 >$ $< n \| \hat{a}^+\hat{a} \| n >= \| c_n \|^2$ $< n \| \hat{n} \| n >= \| c_n \|^2$ $n = \| c_n \|^2$ $n = \| c_n \|^2$ $\sqrt{n} = c_n$
$\boxed{\hat{a}^+ \| n >= \sqrt{n+1} \| n+1 >}$	$\boxed{\hat{a} \| n >= \sqrt{n} \| n-1 >}$

and the only non-vanishing matrix elements for the $\hat{a}, \hat{a}^+$ operators are

$< n+1 \| \hat{a}^+ \| n >= \sqrt{n+1}$	$< n-1 \| \hat{a} \| n >= \sqrt{n}$

$$< n+1|\hat{a}^{+}|n >= \sqrt{n+1}$$

$$\begin{bmatrix} 0 & & & & & & & & & \\ \sqrt{1} & 0 & & & & & & & & \\ & \sqrt{2} & 0 & & & & & & & \\ & & \sqrt{3} & 0 & & & & & & \\ & & & \sqrt{4} & 0 & & & & & \\ & & & & \sqrt{5} & 0 & & & & \\ & & & & & . & 0 & & & \\ & & & & & & . & 0 & & \\ & & & & & & & . & 0 & \\ & & & & & & & & . & 0 \end{bmatrix}$$

$$< n-1|\hat{a}|n >= \sqrt{n}$$

$$\begin{bmatrix} 0 & \sqrt{1} & & & & & & & & \\ & 0 & \sqrt{2} & & & & & & & \\ & & 0 & \sqrt{3} & & & & & & \\ & & & 0 & \sqrt{4} & & & & & \\ & & & & 0 & \sqrt{5} & & & & \\ & & & & & 0 & . & & & \\ & & & & & & 0 & . & & \\ & & & & & & & 0 & . & \\ & & & & & & & & 0 & . \\ & & & & & & & & & 0 \end{bmatrix}$$

n=0, 1, 2…

and some tricky equations:

$$\hat{n}|n >= n\ |n >$$

$$(\hat{n})^m |n >= n^m |n >$$

$$\hat{n}|n-1 >= (n-1)|n-1 >$$

$$(\hat{n})^m |n-1 >= (n-1)^m |n-1 >$$

and… $(\hat{a}^{+})^n |0 >= \sqrt{n!}\,|n >$ $\quad (\hat{a})^n |n >= \sqrt{n!}\,|0 >$

QUANTUM FLUCTUATIONS OF THE SINGLE-MODE FIELD

$$<n|\hat{E}_z|n> = <n|\sqrt{\frac{\hbar\omega}{V\varepsilon_0}}(\hat{a}+\hat{a}^+)\sin(k\cdot y)|n> = A<n|\hat{a}|n> + B<n|\hat{a}^+|n> = 0+0=0$$

but

$$<n|\hat{E}_z^2|n> = \frac{\hbar\omega}{V\varepsilon_0}\sin^2(k\cdot y)\left(<n|\hat{a}^+\hat{a}^+|n> + <n|\hat{a}\hat{a}^+|n> + <n|\hat{a}^+\hat{a}|n> + <n|\hat{a}\ \hat{a}|n>\right) =$$

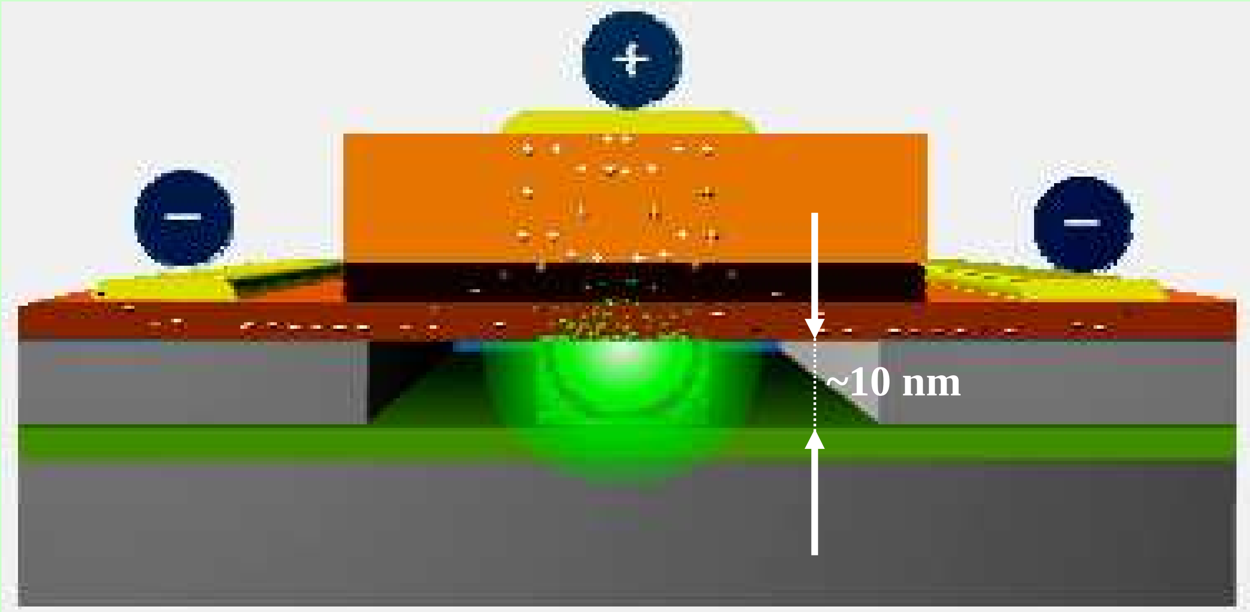
$$= 2\frac{\hbar\omega}{V\varepsilon_0}\left(n+\frac{1}{2}\right)\sin^2(k\cdot y)$$

For example, for a single photon, experimentally detectable electric field equals

$$\boxed{E_z(y,t) = \sqrt{2\frac{\hbar\omega}{V\varepsilon_0}\left(1+\frac{1}{2}\right)}\sin(k\cdot y)}$$

Example:

$$E_z(y,t) = \sqrt{2\frac{\hbar\omega}{V\varepsilon_0}\left(1+\frac{1}{2}\right)}\sin(k\cdot y)$$



http://www.intel.com/research/platform/sp/hybridlaser.htm

$$V \approx 10nm\cdot 10nm\cdot 10nm = 1000nm^3,\ \ \omega = 2\pi f\ = 2\pi c/\lambda$$

$$< n\,|\,\hat{E}_z^2\,|\,n >= \frac{\hbar\omega}{V\varepsilon_0}\left(n+\frac{1}{2}\right) = \frac{\hbar\omega}{V\varepsilon_0}\cdot n + \frac{\hbar\omega}{2V\varepsilon_0} = An + B$$

$$\frac{\hbar\omega}{V\varepsilon_0} = \frac{1.05\cdot 10^{-34}\cdot 6.28\cdot 3\cdot 10^{8}}{10^{3}\cdot 10^{-27}\cdot 8.85\cdot 10^{-12}} = 2.35\cdot 10^{10}\,(V^2/m^2)$$

$$\sqrt{\frac{\hbar\omega}{V\varepsilon_0}} = 1.5\cdot 10^{5}\,V/m$$